11.1 Radical Expressions and Rational Exponents

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Essential Question: How are rational exponents related to radicals and roots?

Explore Defining Rational Exponents in Terms of Roots

Remember that a number a is an nth root of a number b if $a^n = b$. As you know, a square root is indicated by $\sqrt{\ }$ and a cube root by $\sqrt[3]{\ }$. In general, the n^{th} root of a real number a is indicated by $\sqrt[n]{a}$, where n is the **index** of the radical and a is the radicand. (Note that when a number has more than one real root, the radical sign indicates only the principal, or positive, root.)

A *rational exponent* is an exponent that can be expressed as $\frac{m}{n}$, where m is an integer and n is a natural number. You can use the definition of a root and properties of equality and exponents to explore how to express roots using rational exponents.

(A) How can you express a square root using an exponent? That is, if $\sqrt{a} = a^m$, what is m?

Given	$\sqrt{a} = a^m$
Square both sides.	$(\sqrt{a})^2 = (a^m)^2$
Definition of square root	$a = (a^m)^2$
Power of a power property	$a = a^{2m}$
Definition of first power	$a^{1} = a^{2m}$
The bases are the same, so equate exponents.	$1 = 2m$
Solve.	$m = \frac{1}{2}$
So,	$\sqrt{a} = a^{\frac{1}{2}}$.

You are multiplying the exponents not adding

Important

Essential Question: How are rational exponents related to radicals and roots?

Possible answer: Rational exponents and radicals are both ways to represent roots of quantities. The denominator of a rational exponent and the index of a radical represent the root. The rational exponent $\frac{m}{n}$ on a quantity represents the mth power of the nth root of the quantity or the nth root of the mth power of the quantity, where n is the index of the radical.

(B) How can you express a cube root using an exponent? That is, if $\sqrt[3]{a} = a^m$, what is m?

Given	$\sqrt[3]{a} = a^m$
Cube both sides.	$(\sqrt[3]{a})^3 = (a^m)^3$
Definition of cube root	$a = (a^m)^3$
Power of a power property	$a = a^{3m}$
Definition of first power	$a^1 = a^{3m}$
The bases are the same, so equate exponents.	$1 = 3m$
Solve.	$m = \frac{1}{3}$
So,	$\sqrt[3]{a} = a^{\frac{1}{3}}$.

Reflect

1. **Discussion** Examine the reasoning in Steps A and B. Can you apply the same reasoning for any *n*th root, $\sqrt[n]{a}$, where *n* is a natural number? Explain. What can you conclude?
Yes; the only difference is that instead of squaring or cubing both sides and using the definition of square root or cube root, you raise both sides to the *n*th power and use the definition of *n*th root. The other reasoning is exactly the same. You can conclude that finding the $\frac{1}{n}$ power is the same as finding the *n*th root, or that $\sqrt[n]{a} = a^{\frac{1}{n}}$.

2. For a positive number *a*, under what condition on *n* will there be only one real *n*th root? two real *n*th roots? Explain.
When *n* is odd; when *n* is even; for an odd power like x^3 or x^5, every distinct value of *x* gives a unique result, so there is only one number that raised to the power will give the result, or one *n*th root. For example, there is one fifth root of 32 because 2 is the only number whose fifth power is 32. For an even power like x^2 or x^4, there are two values of *x* (opposites of each other) that raised to the power will give the result. For example, there are two fourth roots of 81 because 3^4 and $(-3)^4$ both equal 81.

3. For a negative number *a*, under what condition on *n* will there be no real *n*th roots? one real *n*th root? Explain.
When *n* is even; when *n* is odd; no even power of any number is negative, but every number—positive or negative—has exactly one *n*th root when *n* is odd.

QUESTIONING STRATEGIES

? When rewriting a radical expression by using a rational exponent, where do you place the index of the radical? **in the denominator of the rational exponent**

? How does knowing that $a^{\frac{1}{n}} = \sqrt[n]{a}$ help you to simplify the expression $16^{0.25}$? **You can rewrit the fraction as $\frac{1}{4}$. So $16^{0.25} = 16^{\frac{1}{4}} = \sqrt[4]{16} = 2$.**

AVOID COMMON ERRORS

Students may need to be reminded that although, for example, both 3 and −3 are fourth roots of 81, the expression $\sqrt[4]{81}$ indicates the *positive* (principal) fourth root of 81, or 3. Thus, the expression $81^{\frac{1}{4}}$ simplifies to 3, not to both 3 and −3.

Explain 1 Translating Between Radical Expressions and Rational Exponents

In the Explore, you found that a rational exponent $\frac{m}{n}$ with $m = 1$ represents an nth root, or that $a^{\frac{1}{n}} = \sqrt[n]{a}$ for positive values of a. This is also true for negative values of a when the index is odd. When $m \neq 1$, you can think of the numerator m as the power and the denominator n as the root. The following ways of expressing the exponent $\frac{m}{n}$ are equivalent.

Rational Exponents

For any natural number n, integer m, and real number a when the nth root of a is real:

Words	Numbers	Algebra
The exponent $\frac{m}{n}$ indicates the mth power of the nth root of a quantity.	$27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$	$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
The exponent $\frac{m}{n}$ indicates the nth root of the mth power of a quantity.	$4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$	$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Notice that you can evaluate each example in the "Numbers" column using the equivalent definition.

$$27^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{729} = 9 \qquad 4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$$

Example 1 **Translate radical expressions into expressions with rational exponents, and vice versa. Simplify numerical expressions when possible. Assume all variables are positive.**

(A) **a.** $(-125)^{\frac{4}{3}}$ **b.** $x^{\frac{11}{8}}$ **c.** $\sqrt[5]{6^4}$ **d.** $\sqrt[4]{x^3}$

a. $(-125)^{\frac{4}{3}} = \left(\sqrt[3]{-125}\right)^4 = (-5)^4 = 625$

b. $x^{11/8} = \sqrt[8]{x^{11}}$ or $\left(\sqrt[8]{x}\right)^{11}$

c. $\sqrt[5]{6^4} = 6^{\frac{4}{5}}$

d. $\sqrt[4]{x^3} = x^{\frac{3}{4}}$

(B) **a.** $\left(\frac{81}{16}\right)^{\frac{3}{4}}$ **b.** $(xy)^{\frac{5}{3}}$ **c.** $\sqrt[3]{11^6}$ **d.** $\sqrt[3]{\left(\frac{2x}{y}\right)^5}$

a. $\left(\frac{81}{16}\right)^{\frac{3}{4}} = \left(\sqrt[\square]{\frac{81}{16}}\right)^{\square} = \left(\square\right)^3 = \square$

b. $(xy)^{\frac{5}{3}} = \sqrt[\square]{(xy)^{\square}}$ or $\left(\sqrt[\square]{xy}\right)^{\square}$

c. $\sqrt[3]{11^6} = 11^{\square} = 11^{\square} = \square$

d. $\sqrt[3]{\left(\frac{2x}{y}\right)^5} = \left(\frac{2x}{y}\right)^{\square}$

(B) **a.** $\left(\frac{81}{16}\right)^{\frac{3}{4}}$ **b.** $(xy)^{\frac{5}{3}}$ **c.** $\sqrt[3]{11^6}$ **d.** $\sqrt[3]{\left(\frac{2x}{y}\right)^5}$

a. $\left(\frac{81}{16}\right)^{\frac{3}{4}} = \left(\sqrt[4]{\frac{81}{16}}\right)^{3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

b. $(xy)^{\frac{5}{3}} = \sqrt[3]{(xy)^{5}}$ or $\left(\sqrt[3]{xy}\right)^{5}$

c. $\sqrt[3]{11^6} = 11^{\frac{6}{3}} = 11^{2} = 121$

d. $\sqrt[3]{\left(\frac{2x}{y}\right)^5} = \left(\frac{2x}{y}\right)^{\frac{5}{3}}$

Reflect

4. How can you use a calculator to show that evaluating $0.001728^{\frac{4}{3}}$ as a power of a root and as a root of a power are equivalent methods?

Reflect

4. How can you use a calculator to show that evaluating $0.001728^{\frac{4}{3}}$ as a power of a root and as a root of a power are equivalent methods?

As a power of a root: Enter $\sqrt[3]{0.001728}$ to obtain 0.12. Then enter 0.12^4 to obtain 0.00020736.

As a root of a power: Enter 0.001728^4. Then find $\sqrt[3]{(\text{Ans.})}$. The result is again 0.00020736.

Your Turn

5. Translate radical expressions into expressions with rational exponents, and vice versa. Simplify numerical expressions when possible. Assume all variables are positive.

a. $\left(-\frac{32}{243}\right)^{\frac{2}{5}}$

b. $(3y)^{\frac{b}{c}}$

c. $\sqrt[3]{0.5^9}$

d. $\left(\sqrt[u]{st}\right)^v$

Your Turn

5. Translate radical expressions into expressions with rational exponents, and vice versa. Simplify numerical expressions when possible. Assume all variables are positive.

a. $\left(-\frac{32}{243}\right)^{\frac{2}{5}}$

$\left(-\frac{32}{243}\right)^{\frac{2}{5}} = \left(\sqrt[5]{-\frac{32}{243}}\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}$

b. $(3y)^{\frac{b}{c}}$

$(3y)^{\frac{b}{c}} = \left(\sqrt[c]{3y}\right)^b$ or $\sqrt[c]{(3y)^b}$

c. $\sqrt[3]{0.5^9}$

$\sqrt[3]{0.5^9} = 0.5^{\frac{9}{3}} = 0.5^3 = 0.125$

d. $\left(\sqrt[u]{st}\right)^v$

$\left(\sqrt[u]{st}\right)^v = (st)^{\frac{v}{u}}$

QUESTIONING STRATEGIES

? How is a power function related to a radical function? **It is the same as the related radical function, just a way of expressing the function with a rational exponent instead of a radical.**

? How do you identify the restrictions on the domain of a power function that represents a real-world situation? **The domain must be restricted to numbers that make x^b a real number, and further restricted to numbers that make sense in the context of the situation.**

Explain 2 Modeling with Power Functions

The following functions all involve a given power of a variable.

$A = \pi r^2$ (area of a circle)

$V = \frac{4}{3}\pi r^3$ (volume of a sphere)

$T = 1.11 \cdot L^{\frac{1}{2}}$ (the time T in seconds for a pendulum of length L feet to complete one back-and-forth swing)

These are all examples of *power functions*. A power function has the form $y = ax^b$ where a is a real number and b is a rational number.

Example 2 **Solve each problem by modeling with power functions.**

(A) **Biology** The function $R = 73.3\sqrt[4]{M^3}$, known as Kleiber's law, relates the basal metabolic rate R in Calories per day burned and the body mass M of a mammal in kilograms. The table shows typical body masses for some members of the cat family.

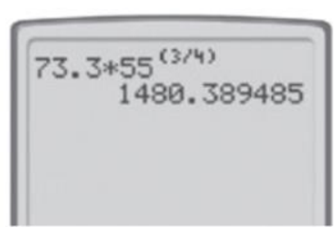
Images/Corbis

Typical Body Mass	
Animal	**Mass (kg)**
House cat	4.5
Cheetah	55
Lion	170

a. Rewrite the formula with a rational exponent.

b. What is the value of R for a cheetah to the nearest 50 Calories?

c. From the table, the mass of the lion is about 38 times that of the house cat. Is the lion's metabolic rate more or less than 38 times the cat's rate? Explain.



- Answers

a. Because $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, $\sqrt[4]{M^3} = M^{\frac{3}{4}}$, so the formula is $R = 73.3M^{\frac{3}{4}}$.

b. Substitute 55 for M in the formula and use a calculator.

The cheetah's metabolic rate is about 1500 Calories.

c. Less; find the ratio of R for the lion to R for the house cat.

$$\frac{73.3(170)^{\frac{3}{4}}}{73.3(4.5)^{\frac{3}{4}}} = \frac{170^{\frac{3}{4}}}{4.5^{\frac{3}{4}}} \approx \frac{47.1}{3.1} \approx 15$$

The metabolic rate for the lion is only about 15 times that of the house cat.

(B) The function $h(m) = 241m^{-\frac{1}{4}}$ models an animal's approximate resting heart rate h in beats per minute given its mass m in kilograms.

a. A common shrew has a mass of only about 0.01 kg. To the nearest 10, what is the model's estimate for this shrew's resting heart rate?

b. What is the model's estimate for the resting heart rate of an American elk with a mass of 300 kg?

c. Two animal species differ in mass by a multiple of 10. According to the model, about what percent of the smaller animal's resting heart rate would you expect the larger animal's resting heart rate to be?

a. Substitute ______ for m in the formula and use a calculator.

$$h(m) = 241\left(\square\right)^{-\frac{1}{4}} \approx \square$$

The model estimates the shrew's resting heart rate to be about ______ beats per minute.

b. Substitute ______ for m in the formula and use a calculator.

$$h(m) = 241\left(\square\right)^{-\frac{1}{4}} \approx \square$$

The model estimates the elk's resting heart rate to be about _____ beats per minute.

c. Find the ratio of $h(m)$ for the ________ animal to the ________ animal. Let 1 represent the mass of the smaller animal.

$$\frac{241 \cdot \square^{-\frac{1}{4}}}{241 \cdot 1^{-\frac{1}{4}}} = \square^{-\frac{1}{4}} = \frac{1}{10^{\square}} \approx \square$$

You would expect the larger animal's resting heart rate to be about ______ of the smaller animal's resting heart rate

- **Answers**

a. Substitute 0.01 for m in the formula and use a calculator.

$$h(m) = 241\left(0.01\right)^{-\frac{1}{4}} \approx 760$$

The model estimates the shrew's resting heart rate to be about 760 beats per minute.

b. Substitute 300 for m in the formula and use a calculator.

$$h(m) = 241\left(300\right)^{-\frac{1}{4}} \approx 60$$

The model estimates the elk's resting heart rate to be about 60 beats per minute.

c. Find the ratio of $h(m)$ for the larger animal to the smaller animal. Let 1 represent the mass of the smaller animal.

$$\frac{241 \cdot 10^{-\frac{1}{4}}}{241 \cdot 1^{-\frac{1}{4}}} = 10^{-\frac{1}{4}} = \frac{1}{10^{\frac{1}{4}}} \approx 0.56$$

You would expect the larger animal's resting heart rate to be about 56% of the smaller animal's resting heart rate.

Reflect

6. What is the difference between a power function and an exponential function?

7. In Part B, the exponent is negative. Are the results consistent with the meaning of a negative exponent that you learned for integers? Explain.

Reflect

6. What is the difference between a power function and an exponential function?

A power function involves a given power of a variable, while an exponential function involves a variable power of a given number (the base).

7. In Part B, the exponent is negative. Are the results consistent with the meaning of a negative exponent that you learned for integers? Explain.

Yes; a power with a negative integer exponent is the reciprocal of the corresponding positive power. So, for example, for the elk, this would mean that $300^{-\frac{1}{4}} = \frac{1}{300^{\frac{1}{4}}}$. Using the calculator again, $h(m) = 241\left(\frac{1}{300^{\frac{1}{4}}}\right) \approx 60$, which is consistent.

Your Turn

8. Use Kleiber's law from Part A.

a. Find the basal metabolic rate for a 170 kilogram lion to the nearest 50 Calories. Then find the formula's prediction for a 70 kilogram human.

b. Use your metabolic rate result for the lion to find what the basal metabolic rate for a 70 kilogram human *would* be *if* metabolic rate and mass were directly proportional. Compare the result to the result from Part a.

Your Turn

8. Use Kleiber's law from Part A.

a. Find the basal metabolic rate for a 170 kilogram lion to the nearest 50 Calories. Then find the formula's prediction for a 70 kilogram human.

Kleiber's law for lion: $73.3(170)^{\frac{3}{4}} \approx 3450$ Calories

Kleiber's law for human: $73.3(70)^{\frac{3}{4}} \approx 1750$ Calories

b. Use your metabolic rate result for the lion to find what the basal metabolic rate for a 70 kilogram human *would* be *if* metabolic rate and mass were directly proportional. Compare the result to the result from Part a.

If metabolic rate and mass were directly proportional then

$\frac{3450 \text{ Cal}}{170 \text{ kg}} = \frac{x \text{ Cal}}{70 \text{ kg}}$, so $170x = (3450)(70)$, or $x = \left(\frac{241{,}500}{170}\right) \approx 1400$ Cal.

so, the rate for a human would be significantly lower than the actual prediction from Kleiber's law. Kleiber's law indicates that smaller organisms have a higher metabolic rate per kilogram of mass than do larger organisms.

Elaborate

9. Explain how can you use a radical to write and evaluate the power $4^{2.5}$.

10. When $y = kx$ for some constant k, y varies directly as x. When $y = kx^2$, y varies directly as the square of x; and when $y = k\sqrt{x}$, y varies directly as the square root of x. How could you express the relationship $y = kx^{\frac{3}{5}}$ for a constant k?

11. **Essential Question Check-In** Which of the following are true? Explain.

- To evaluate an expression of the form $a^{\frac{m}{n}}$, first find the nth root of a. Then raise the result to the mth power.
- To evaluate an expression of the form $a^{\frac{m}{n}}$, first find the mth power of a. Then find the nth root of the result.

- Answers

Elaborate

9. Explain how can you use a radical to write and evaluate the power $4^{2.5}$.

You can first rewrite the decimal as the fraction $\frac{25}{10} = \frac{5}{2}$. Then $4^{2.5} = 4^{\frac{5}{2}} = \left(\sqrt{4}\right)^5 = 2^5 = 32$.

10. When $y = kx$ for some constant k, y varies directly as x. When $y = kx^2$, y varies directly as the square of x; and when $y = k\sqrt{x}$, y varies directly as the square root of x. How could you express the relationship $y = kx^{\frac{3}{5}}$ for a constant k?

***y* varies directly as the three-fifths power of *x*.**

11. **Essential Question Check-In** Which of the following are true? Explain.

- To evaluate an expression of the form $a^{\frac{m}{n}}$, first find the nth root of a. Then raise the result to the mth power.
- To evaluate an expression of the form $a^{\frac{m}{n}}$, first find the mth power of a. Then find the nth root of the result.

They are both true. For a real number a and integers m and n with $n \neq 0$, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$, so the order in which you find the root or power does not matter.

SUMMARIZE THE LESSON

How can you rewrite a radical expression as an exponential expression and vice versa? **You can write a radical expression as the radicand raised to a fraction in which the numerator is the power of the radicand and the denominator is the index of the radical. You can write an exponential expression with the base of the exponent as the radicand, the denominator of the exponent as the index, and the numerator of the exponent as the power.**

End of Lesson : Please do the classwork and homework of 11.1 online as done during the last quarter. Go to your student portal---click on Algerba 2 ---the math book and do the online work by the due date by 7:00 am
Any questions email:
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